

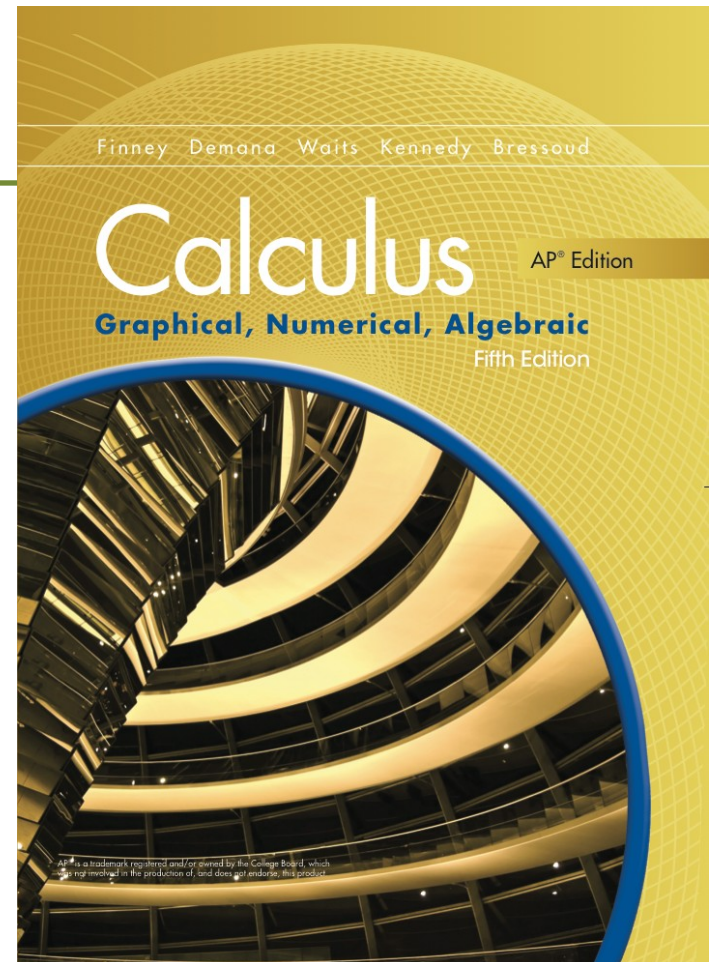
Chapter 8

Applications of Definite Integral



Section 8.1

Accumulation and Net Change



Quick Review

Find all values of x (if any) at which the function changes sign on the given interval.

1. $\cos 2x$ on $[0, 1]$

2. $x^2 - 5x + 6$ on $[-5, 5]$

3. e^{-x} on $[0, \infty)$

4. $\frac{x^2 - 1}{x^2 - 4}$ on $[-5, 5]$

Quick Review Solutions

Find all values of x (if any) at which the function changes sign on the given interval.

1. $\cos 2x$ on $[0, 1]$ $\frac{\pi}{4}$

2. $x^2 - 5x + 6$ on $[-5, 5]$ $2, 3$

3. e^x on $[0, \infty)$ always positive

4. $\frac{x^2 - 1}{x^2 - 4}$ on $[-5, 5]$ $-2, -1, 1, 2$

What you'll learn about

- Displacement as integral of velocity
- Total distance as integral of absolute value of velocity
- Modeling accumulation using Riemann sum approximations
- Interpretation of limits of Riemann sums as definite integrals
- Finding net change given graphical and/or tabular representations of rate of change

...and why

The integral is a tool that can be used to calculate net change and total accumulation.

Example Linear Motion Revisited

$v(t) = 10 - 2t$ is the velocity in m/sec of a particle moving along the x -axis when $0 \leq t \leq 9$. Use analytic methods to:

- (a)** Determine when the particle is moving to the right, to the left, and stopped.
- (b)** Find the particle's displacement for the given time interval.
- (c)** If $s(0) = 3$, what is the particle's final position?
- (d)** Find the total distance traveled by the particle.

(a) When $v(t) > 0$, the particle is moving right. This occurs when $0 \leq t < 5$. When $v(t) = 0$, the particle is stopped. This occurs when $t = 5$. When $v(t) < 0$, the particle is moving left. This occurs when $5 < t \leq 9$.

Example Linear Motion Revisited

$$\begin{aligned}\text{(b) displacement} &= \int_0^9 v(t) dt \\ &= \int_0^9 (10 - 2t) dt \\ &= \left[10t - t^2 \right]_0^9 \\ &= 90 - 81 \\ &= 9\end{aligned}$$

During the first 9 seconds of motion, the particle moves 9 m to the right.

(c) It starts at $s(0) = 3$, so its position at $t = 9$ is

$$\begin{aligned}\text{New position} &= \text{initial position} + \text{displacement} \\ &= 3 + 9.\end{aligned}$$

Example Linear Motion Revisited

$$\begin{aligned} \text{(d) Total distance} &= \int_0^9 |v(t)| dt = \int_0^9 |10 - 2t| dt \\ &= \int_0^5 (10 - 2t) dt + \int_5^9 (2t - 10) dt \\ &= \left[10t - t^2 \right]_0^5 + \left[t^2 - 10t \right]_5^9 \\ &= 50 - 25 + 81 - 90 - 25 + 50 = 41 \text{ m} \end{aligned}$$

Strategy for Modeling with Integrals

1. Approximate what you want to find as a Riemann sum of values of a continuous function multiplied by interval lengths. If $f(x)$ is the function and $[a, b]$ the interval, and you partition the interval into subintervals of length Δx , the approximating sums will have the form $\sum f(c_k)\Delta x$ with c_k a point in the k th subinterval.
2. Write a definite integral, here $\int_a^b f(x)dx$, to express the limit of these sums as the norms of the partitions go to zero.
3. Evaluate the integral numerically or with an antiderivative.

Consumption over Time

The integral is a natural tool to calculate net change and total **accumulation** of more quantities than just distance and velocity. Integrals can be used to calculate growth, decay, and, as in the next example, consumption. Whenever you want to find the cumulative effect of a varying rate of change, integrate it.

Example Potato Consumption

From 1970 to 1980, the rate of potato consumption in a particular country was $C(t) = 2.2 + 1.1^t$ millions of bushels per year, with t being years since the beginning of 1970. How many bushels were consumed from the beginning of 1972 to the end of 1975?

We seek the cumulative effect of the consumption rate for $2 \leq t \leq 6$.

Step 1: Riemann sum: We partition $[2, 6]$ into subintervals of length Δt and let t_k be a time in the k th subinterval. The amount consumed during this interval is approximately $C(t_k) \Delta t$ million bushels.

The consumption for $2 \leq t \leq 6$ is approximately $\sum C(t_k) \Delta t$ million bushels.

Example Potato Consumption

Step 2: Definite integral: The amount consumed from $t = 2$ to $t = 6$ is the limit of these sums as the norms of the partitions

go to zero $\int_2^6 C(t) dt = \int_2^6 (2.2 + 1.1^t) dt$ million bushels.

Step 3: Evaluate: Evaluating numerically, we obtain

$\text{NINT}(2.2 + 1.1^t, t, 2, 6) \approx 14.692$ million bushels.

Work

When a body moves a distance d along a straight line as a result of the action of a force of constant magnitude F in the direction of motion, the **work** done by the force is

$$W = Fd.$$

The equation $W = Fd$ is the **constant - force formula** for work.

The units of work are force $\times$ distance. In the metric system, the unit is the **newton - meter** or **joule**. In the U.S. customary system, the most common unit is the **foot - pound**.

Hooke's Law

Hooke's Law for springs says that the force it takes to stretch or compress a spring x units from its natural length is a constant times x . In symbols: $F = kx$, where k , measured in force units per length, is a **force constant**.

Example A Bit of Work

It takes a force of 6N to stretch a spring 2 m beyond its natural length.

How much work is done in stretching the spring 5 m from its natural length?

By Hooke's Law, $F(x) = kx$. We know that $F(2) = 6 = 2k$, so $k = 3$ N/m and $F(x) = 3x$ for this spring.

Step 1: Riemann sums: We partition the interval into subintervals on each of which F is so nearly constant that we can apply the constant-force formula for work. If x_k is any point in the k th subinterval, the value of F throughout the interval is approximately $F(x_k) = 3x_k$. The work done across the interval is approximately $3x_k \Delta x$, where Δx is the length of the interval. The sum $\sum F(x_k) \Delta x = \sum 3x_k \Delta x$ approximates the work done by F from $x=0$ to $x=5$.

Example A Bit of Work

Steps 2 and 3: Integrate: The limit of these sums as the norm of the partitions go to zero is

$$\begin{aligned}\int_0^5 F(x) dx &= \int_0^5 3x dx \\ &= \left. \frac{3}{2} x^2 \right|_0^5 \\ &= \frac{75}{2} \text{ N} \cdot \text{m}.\end{aligned}$$